Motor Driven Motion

If we have a motor force $F(x,v)$ defined by a linear force-velocity trade-off over a constant range of motion, d, by

$$F(x,v) = \begin{cases} F_{max}(1 - v/v_{max}) & \text{for } 0 \le x \le d \\ 0 & \text{otherwise} \end{cases}$$

then we can solve for the take-off velocity $v_{to} = v(x=d)$ using Newton's 2nd law:

$$F = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{F_{max}}{m}\left(1 - \frac{v}{v_{max}}\right) = \frac{F_{max}}{m}\left(\frac{v_{max} - v}{v_{max}}\right)$$

using the chain rule, $\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot v$

$$\Rightarrow \frac{dv}{dx} = \frac{F_{max}}{m}\frac{(v_{max} - v)}{v_{max}\, v}$$

using separation of variables and integrating from $v\big|_0^{v_{to}}$ and $x\big|_0^{d}$

$$\int_0^{v_{to}} \frac{v_{max}\, v}{v_{max} - v}\, dv = \frac{F_{max}}{m}\int_0^d dx$$

let $U = v_{max} - v$, $dU = -dv$

$v = v_{max} - U$

$$\Rightarrow \int_{v_{max}}^{v_{max}-v_{to}} \left(\frac{v_{max}}{U} - 1\right) dU = -\frac{F_{max}\, d}{m\, v_{max}}$$

$$v_{max}\cdot \ln U\Big|_{v_{max}}^{v_{max}-v_{to}} - U\Big|_{v_{max}}^{v_{max}-v_{to}} = -\frac{F_{max}\, d}{m\, v_{max}}$$

$$\ln\left(\frac{v_{max} - v_{to}}{v_{max}}\right) + \frac{v_{to}}{v_{max}} = -\frac{F_{max}\, d}{m v_{max}^2}$$

$$\frac{v_{to}}{v_{max}} + \ln\left(1 - \frac{v_{to}}{v_{max}}\right) = -\frac{F_{max}\, d}{m v_{max}^2}$$ (*) transcendental equation describing $v_{to}(m)$